Comparative Assessment of Two Estimates of the Bernoulli Distribution Parameters in Analysis of Short-Term Rainfall Intensity

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ORIGINAL RESEARCH

Abstract- This paper presents an application ofthe maximum likelihood methodand the Bernoulli distribution of selected rainfall intensity data. The parameter of the density of Bernoullidistributionwas estimated by the maximum likelihood method (MLM), and Microsoft Excel Solver (MES). The calculated probabilities using the estimated parameter were evaluated statistically (analysis of variance (ANOVA), relative error, model of' selection criterion (MSC), Coefficient of Determination (CD) andCorrelation coefficient (R). The study revealed thatthe Bernoulli probability distribution's parameter (p) is the mean of the natural logarithm of rainfall intensity using the MLM estimator. The parameters were 0.665 and 0.535 for Makurdi, 0.695 and 0.478for Abeokuta using MLM and MES, respectively. The relative errors were 0.479 and 0.743, and 1.141 and 1.509 for Makurdi and Abeokuta using MLM and MES, respectively. The correlation coefficients for Makurdi and Abeokuta using MLM and MES were 0.876 and 0.800, and 0.269 and 0.341, respectively.It was concluded thatthe MLM parameter was better than MES based on the values of MSC, CD, relative error and R. MLM predicted Weibull probability of rainfall intensitybetter than MES.

Keywords- Bernoulli distribution, Rainfall intensity, Maximum likelihood method, Analysis of variance, Probability density

1 Introduction

Extreme rainfalls can meaningfully affect, socio-economic activities, travel patterns and demand, rainfall-driving products, water demand and consumption, behaviour and traffic (air, land and water) movement characteristics (Akin *et al.,* 2011). The Rainfall Intensity Duration Frequency and probability distributions relationship are the most frequently used implements for the design and assessment of hydraulic, hydrological, water resources and engineering control infrastructures (David *et al.,* 2019). Rainfall Intensity Duration Frequency expressions are mathematical and statistical equations that present the relationship between the rainfall intensity, duration and frequencies (return periods). describe all the possible values and likelihoods that a random variable can occur within a specific range mathematical expressions or functions that (Couton*et al.,* 1997; Balakrishnan *et al.,* 2014; Baillon *et al.,* 2015; Bianchi *et al.,* 2016; Du and Xie 2020).

The combined influence of rainfall intensity and duration on flood (maximum discharges of runoff) from urban watersheds has been well documented (Al-Amri and Subyani, 2017; Al-Zahran, 2018; Kwak *et al.,* 2020). Figure 1 (a to d) shows the effect of floods on the environment. Probability distributions are idealized frequency distributions**.** These probability distributions are statistical or number of principles of sample data concordance with the normal probability distribution law have been categorized, and some approvals for using these criteria in the practice of engineering statistics, environmental science and management analysis have been established (Aleksandrovskaya*et al.,* 2019). These probability distributions are divided into two parts as follows (Oke, 2008; Oke *et al.,* 2008; Fiondella and Zeephonsekul, 2015; Gao *et al.,* 2019; Jones *et al.,* 2020):

a) Discrete Probability Distributions (Binomial Distribution, Bernoulli Distribution and Poisson Distribution)
b) Continuous Probability Distributions (Normal Distribution, Continuous Uniform Distribution, Log-Normal Distribution, Exponential Distribution, Gamma, Pearson Type III, log- Pearson Type III, Gumbel, Weibull, Chin, 2013)

The binomial distributions are discrete distributions with a finite number of possibilities. When observing a series of what are known as Bernoulli trials, the binomial distribution emerges. A Bernoulli trial is a scientific experiment with only two outcomes: success or failure (Kadane, 2016; Kawakami *et al.,* 2017; Kawamura, 1988; Kazuki *et al.,* 2017; Kullback 1935, Lidiya *et al.,* 2019; Luo *et al.,2017;* 2019; Molotkov, 2018). There are several methods of estimation to estimate the reliability of these probability distributions. These methods include Maximum likelihood estimation, least square and weighted least square estimation, Percentile estimation, Maximum product of estimation, Minimum spacing distance estimation, Cramer-Von Mises estimation, Anderson-Darling and Right-tail Anderson-Darling estimation (Almarashi *et al.,* 2020). More data on applications and information on Bernoulli distribution and other probability distributions are well documented in literature such as Brousius (2015); Brun and Melkote (2012); Calisto and Bologna (2007); Chai *et al.,* (2020); Dai *et al.,* (2013); Denuit and Vernic (2018); Gharib *et al.,* (2014); Gharib (2014); Huber (2016); Jeong and Yon (2020) and Jeong *et al.,* (2016).

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Section E- CIVIL ENGINEERING AND RELATED SCIENCES
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Fig. 1a: Aerial View of typical Effect of Floods in Community

Fig. 1b: Collapsed Building due to a typical Effect of Floods in Community

Fig. 1c: Aerial View of Flooded areas in Community

Fig. 1d: Aerial View of Flooded Road

Literature provides information on probability distributions (normal, and log-normal), but there is little (Pichugian, 2008 described normal and gamma distribution of river runoff; Baillon *et al.*, 2015 utilized Bernoulli distribution for sharp uniform bound; Barneet et al., 2016 used Binomial and Weibull distribution for yielding and reliability prediction I electrical materials; Denuit and Vernic, 2018 provide information on bivariate Bernoulli weighted sums; Pham and Pham, 2019 established information on a median -based machine - learning approach for predicting random sampling Bernoulli distribution parameter) or no information on the application of Maximum likelihood estimation and Bernoulli distribution for rainfall intensity data (Chai *et al.*, 2020 used likelihood for signal processing, Du and Xie, 2020 utilized computer technologies, which makes it possible to collect real-time rainfall intensity-duration data at various stations there is then the need to utilize Maximum likelihood estimation and Bernoulli distribution for rainfall intensity-duration data analysis. This study, therefore focuses on the utilization and evaluation of Maximum likelihood estimation and Bernoulli distribution for rainfall intensity-duration data analysis.

2 Materials and Method

Rainfall intensity-duration data of two stations (Abeokuta (1986 to 2010) and Makurdi (1979to 2009)) were collected from literature namely David *et al.* (2019) and Isikwue *et al.* (2012). The data were analysed statistically using analysis of variance (ANOVA). Figure 2 presents rainfall intensity data from David *et al.* (2019), while Figure 3 shows rainfall intensity data from Isikwue *et al.* (2012). From the Figure, the highest rainfall-duration-intensity frequency occurred when the duration time was 5 minutes in the year 1 (1979), Isikwue *et al.*, 2012 and 1986, David *et al.*, 2019 respectively) and the lowest rainfall-duration intensity frequency occurred when the duration was 1440 minutes in the 30th year (2009), Isikwue *et al.*,2012 and 2010, David *et al.*, 2019).The probability of the rainfall intensity was computed using the Weibull probability mathematical expression as follows(Teyabeen*et al.*, 2017; Almarashi *et al.*, 2020equations 1 and 2):

$$T_m(x) = \frac{n+1}{m} \quad (1)$$

Where; T_m is the return period, n is the sample size and m is the rank.

$$f(x) = p_m(x) = \frac{1}{T_m} \quad (2)$$

Where; p_m (x) is the theoretical probability (probability index) and f(x) is the cumulative probability

The Weibull distribution is among the most commonly use modelling and distribution for the rainfall intensity data. It was used as a standard to evaluate the performance of MLM, MES and Bernoulli distribution. The parameter of the Bernoulli distribution was calculated using the maximum likelihood estimate method (MLM) and Microsoft Excel Solver (MES). Figure 4 presents the summary of the Microsoft Excel Solver procedures. MES was used for the determination of these empirically derived parameters based on availability at no additional cost. The procedure used for the Microsoft Excel solver can be summarized as follows:

Excel solver was added in Microsoft Excel, the target of numerical analysis operation and changing cells were set,

$$\left(\left(K_p - K_t\right)^2 = 0\right),$$

Where; K_p is the probability value using Weibull mathematical expression

$$\left(f(x) = p_m(x) = \frac{1}{T_m}\right)$$

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and K_t is the Bernoulli distribution probability calculated using MLM

$$\left(f(x) = \quad p^x(1-p)^{(1-x)}\right);$$

and Microsoft Excel Solver was allowed to iterate at 200 iterations with 0.005 tolerance

2.1 Derivation of Bernoulli Parameter using Maximum Likelihood Method

The log-likelihood function of this random sample is given as follows (Couton *et al.*, 1997, Brosius, 2015; Kazuki *et al.*, 2017, equation 3):

$$L(x_1, x_2, x_3, x_4, x_5, ---------- x_n) = \sum_{i=1}^{n} ln\, f\,(x_i, \theta) \quad (3)$$

X_1, X_2,, X_n are random samples of the size of n from the distribution with the probability density function f (x, θ), where θ (θ_1, θ_2,..., θ_k), θ ϵ θ, is the unknown parameter. θ is in general vector parameter and let x_1, x_2,, x_n be a realization of the random sample. The maximum likelihood estimates θ of the parameter θ are the values of θ that maximize (1) with respect to θ. Maximum Likelihood Estimation is a systematic technique for estimating parameters in a probability model from a data sample. Suppose a sample X_1, ..., X_n has been obtained from a probability model specified by mass or density function f (x; θ) depending on parameter(s) θ lying in parameter space θ. The maximum likelihood estimate is produced as follows (Pichugina, 2008; Pobočíková *et al.*, 2017):

Write down the likelihood function (that is, the product of the n mass or density function terms; where the i^{th} term is the mass or density function evaluated at x_i) viewed as a function of θ., L(θ), as Equations (4, 5 and 6):

$$L(\theta) = \quad \prod_{i=1}^{n} fx(x_i, \theta) \quad (4)$$

Take the natural log of the likelihood, collect terms involving θ

$$ln\big(L(\theta)\big) = \quad ln[\prod_{i=1}^{n} fx(x_i, \theta)] \quad (5)$$

Find the value of θ ∈ θ, θ, for which log L(θ) is maximized, for example by differentiation. If θ is a single parameter, find θ by solving

$$\frac{d}{d\theta}\left[ln\big(L(\theta)\big)\right] = \quad \frac{d}{d\theta}\{ln[\prod_{i=1}^{n} fx(x_i, \theta)]\} \quad (6)$$

In the parameter space θ. If θ is vector-valued, say θ = (θ_i, ..., θ_n), then find θ = (θ_i, ..., θ_n) by simultaneously solving the n equations given by equation (7)

$$\frac{\partial}{\partial \theta_j}\left[ln\big(L(\theta)\big)\right] = \quad \frac{\partial}{\partial \theta_j}\left\{ln\left[\prod_{i=1}^{n} fx(x_i, \theta)\right]\right\} \quad = 0 \quad ; \quad j = 1 \dots k \quad (7)$$

In parameter space θ. Note that, if parameter space θ is a bounded interval, then the maximum likelihood estimate may lie on the boundary of θ. Bernoulli probability distribution can be expressed as follows (Yashunsky, 2019; Wentzel and Anhoj, 2019; Parisa *et al.*, 2020; Pham and Pham, 2019; Piast and Piast, 2019; Picho, 2018; Roberts, 2019; Santos, 2018; Sullbhewar and Raveendranath, 2017; Telles, 2020, equations 8 to 17):

$$f(x) = \quad p^x(1-p)^{(1-x)} \quad (8)$$

$$L(x) = \quad \prod_{i=1}^{n} fx(x_i, p) = \quad p^{nx}(1-p)^{n(1-x)} \quad (9)$$

$$ln[L(x)] = \quad ln[\prod_{i=1}^{n} fx(x_i, p)] = \quad nx\, ln(p) + (n - nx)\, ln(1-p) \quad (10)$$

$$\frac{d}{dp}\{ln[L(x)]\} = \quad \frac{d}{dp}\{ln[\prod_{i=1}^{n} fx(x_i, p)]\} = \quad \frac{d}{dp}[nx\, ln(p) + (n - nx)\, ln(1-p)] \quad (11)$$

$$\frac{d}{dp}[nx\, ln(p) + (n - nx)\, ln(1-p)] = \quad \frac{d}{dp}[\sum_{i=1}^{n} x_i\, ln(p) + n\, ln(1-p) - \sum_{i=1}^{n} x_i\, ln(1-p)] \quad (12)$$

$$\frac{d}{dp}[\sum_{i=1}^{n} x_i\, ln(p) + n\, ln(1-p) - \sum_{i=1}^{n} x_i\, ln(1-p)] = \frac{1}{p}\sum_{i=1}^{n} x_i + \frac{n}{1-p} - \frac{1}{(1-p)}\sum_{i=1}^{n} x_i \quad (13)$$

$$\frac{1}{p}\sum_{i=1}^{n} x_i + \frac{n}{1-p} - \frac{1}{(1-p)}\sum_{i=1}^{n} x_i = \quad (1-p)\sum_{i=1}^{n} x_i + np + p\sum_{i=1}^{n} x_i = 0 \quad (14)$$

$$(1-p)\sum_{i=1}^{n} x_i + np + p\sum_{i=1}^{n} x_i = \quad \sum_{i=1}^{n} x_i - np = 0 \quad (15)$$

$$np - \sum_{i=1}^{n} x_i = \quad 0 \quad (16)$$

$$p = \quad \frac{1}{n}\sum_{i=1}^{n} x_i \quad (17)$$

Equation (17) revealed that Bernoulli probability distribution's parameter (p) is the mean of the natural logarithm of rainfall intensity. Table 2, Figures 5 and 6 present values of Bernoulli probability distribution's parameter obtained using MLM and MES, and performance of these methods compared with standard Weibull method. The calculated Bernoulli distribution's parameter (MLM and MES methods) was used to establish the probability distributions. The parameters obtained and the determined probabilities were evaluated statistically using analysis of variance ANOVA), Relative error, Model of selection criterion (MSC), Coefficient of Determination (CD) and Correlation coefficient (R). MSC indicates higher accuracy, validity and a good fit of the methods. MSC was computed using equation (18):

$$MSC \quad = \quad ln\left(\frac{\sum_{i=1}^{n}\left(Y_{obsi} - \overline{Y}_{obs}\right)^2}{\sum_{i=1}^{n}\left(Y_{obsi} - Y_{cali}\right)^2}\right) - \frac{2p}{n} \quad (18)$$

Where; Y_{obsi} is the probability value using the Weibull probability mathematical expression; $\overline{Y}_{obs}$ is the average probability value using the Weibull probability mathematical expression; p is the total number of fixed parameters to be estimated in the methods; n is the total number of rainfall intensities calculated, and Y_{cali} is the probability calculated using the MLM estimator. The coefficient of determination (CD) can be interpreted as the proportion of expected data variation that can be explained by the obtained data. Higher values of CD indicate higher accuracy, validity and good fitness of the device. CD, correlation coefficient and relative error can

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be expressed as equations (19 to 21):

$$CD = \frac{\sum_{i=1}^{n}\left(Y_{obsi}-\overline{Y_{cali}}\right)^{2}-\sum_{i=1}^{n}\left(Y_{obsi}-Y_{cali}\right)^{2}}{\sum_{i=1}^{n}\left(Y_{obsi}-\overline{Y_{cali}}\right)^{2}} \quad (19)$$

Where; $\overline{Y}_{\text{cali}}$ is the average probability value calculated using the MLM estimator.

$$R = \sqrt{\frac{\sum_{i=1}^{n}\left(Y_{obsi}-\overline{Y_{cali}}\right)^{2}-\sum_{i=1}^{n}\left(Y_{obsi}-Y_{cali}\right)^{2}}{\sum_{i=1}^{n}\left(Y_{obsi}-\overline{Y_{cali}}\right)^{2}}} \quad (20)$$

$$R_{el}(\%) = \left(\frac{1}{N}\right)\sum_{i=1}^{N}\left(\frac{\left(Y_{obsi}-Y_{cali}\right)}{Y_{obsi}}\right) \quad (21)$$

3 Results and Discussion

Table 1 shows the result of an ANOVA of the rainfall-duration intensity frequency (Makurdi). From the Table, the $F_{34,\,170}$= 7.59 and p = 4.40 x 10 $^{-20}$ for analysis of the rainfall-duration intensity frequency between the years. This result revealed that there were significant differences between rainfall-duration-intensity frequency values within the years at a 95 % confidence level ($p < 0.05$). In the same Table 1, the results of an ANOVA of rainfall-duration-intensity frequency within the return period of the rainfall. The Table shows that the $F_{5,\,170}$= 253.12 and p = 8.70 x 10 $^{-77}$ for analysis of the rainfall-duration-intensity frequency between the duration of the rainfall. This result revealed that there was a significant difference between rainfall-duration-intensity frequency values within these durations at a 95 % confidence level ($p < 0.05$). Table 1 shows the result of an ANOVA of the rainfall-duration intensity frequency (Abeokuta).

From the Table, the $F_{24,\,288}$ = 11.79 and p = 1.68 x 10 $^{-30}$ for analysis of the rainfall-duration intensity frequency between the years. This result revealed that there were significant differences between rainfall-duration-intensity frequency values within the years at a 95 % confidence level ($p < 0.05$). In the same Table 1, the results of an ANOVA of rainfall-duration-intensity frequency within the duration period of the rainfall. The Table shows that the $F_{12,\,288}$ = 154.40 and p = 9.90 x 10 $^{-118}$ for analysis of the rainfall-duration-intensity frequency between the duration of the rainfall. This result revealed that there was a significant difference between rainfall-duration-intensity frequency values within these durations at a 95 % confidence level ($p < 0.05$). Table 2 presents the values of the Bernoulli's parameter using the estimators. The Table revealed that the value of the parameter was between 0.478 and 0.695 for both MES and MLM estimator methods. These values of the parameter are similar to the values obtained in literature such as Chacko and Mohan (2018) and Gaba *et al.* (2005). Tables 3 and 4 provide information on statistical analysis (ANOVA) of the parameters. Results of the ANOVA for these parameters (Table 3) revealed that there was a significant difference between these values the parameter obtained using the two estimators and methods at a 95 % confidence level ($F_{1,2}$ = 29.06386 and p = 0.032727, which is less than 0.05). Table 4 provide information on statistical analysis (ANOVA) of the performance evaluations of the two methods. Results of the ANOVA for these parameters (Table 4) revealed that there was no significant difference between these values the two estimators (methods) at a 95 % confidence level ($F_{1,2}$ = 0.01211 and p = 0.92242, which is greater than 0.05).

Figures 5 and 6 established that the Bernoulli distribution is a discrete distribution as the probability discontinue between certain rainfall intensity for both Abeokuta and Makurdi data. The figures revealed further that between 104.54 mm/h and 124.82 mm/h the probability of the rainfall intensity was discontinued for the estimator using the MES method. These lower performances of this parameter by MES is similar to the performance of negative binomial distribution (Gaba *et al.*, 2005, Bernet *et al.*, 2006 and Rinne (2009). In addition, the lower performance of the MES method can be attributed to the weak relationship between the Weibull probability and Bernoulli distribution (Eggermont and Laricca, 2009; Ward and Ahlguist, 2018).Tables 5 and 6 present the performance evaluation of the two methods and statistical evaluation, respectively.The table 6 revealed that the relative error was between 0.479 and 1.509, MSC was between -0.553 and 1.427, CD was between -0.073 and 0.768 and R was between 0.269 and 0.876. From these values of relative errors, MSC, CD and R, MLM predicted the Weibull probability better than the MES, based on lower error and higher MSC, CD and R.. Table 6 presents the results of ANOVA conducted on the statistical evaluation. The Table revealed that there was a significant difference between these statistical values obtained using the two methods at a 95 % confidence level ($F_{7,8}$ = 9.705448 and p = 0.00232, which is less than 0.05).

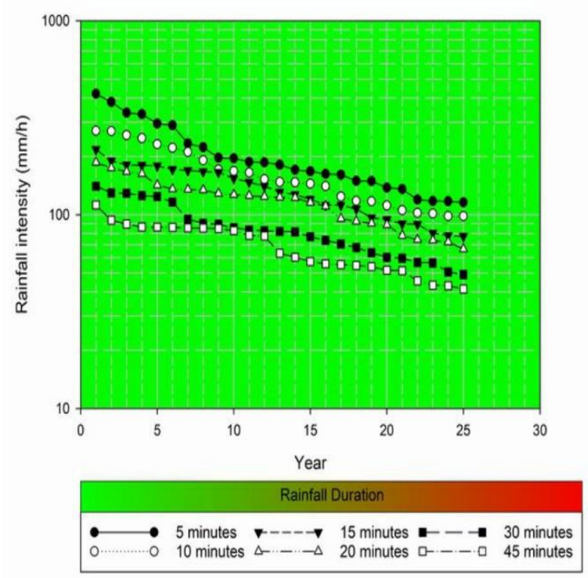

Fig. 2: Rainfall intensity of Abeokuta (duration of between 5 and 45 minutes)

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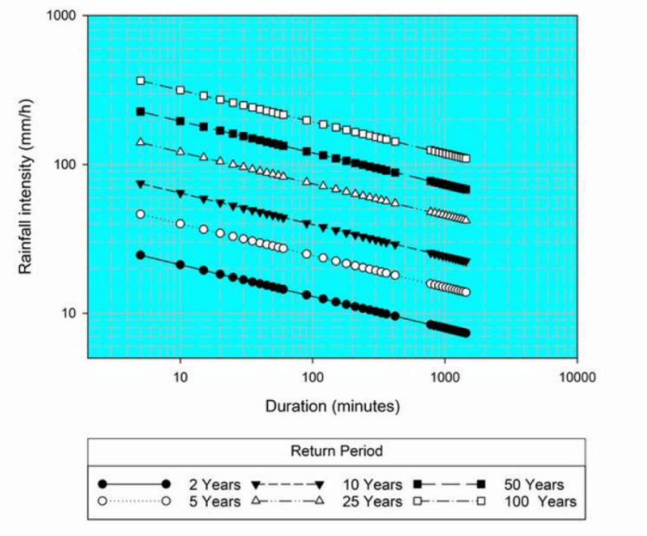

Fig. 3: Rainfall intensity of Makurdi (return period of between 2 and 100 years)

Table 1. Result of ANOVA for Rainfall intensity of Abeokuta

Source of Variation	Sum of Square	Degree of freedom	Mean Sum of Square	F-Value	P-value
Within the Year	181955	24	7581.458	11.79	1.68 x 10^{-30}
Between the Duration	1191745	12	99312.08	154.40	9.9 x 10^{-118}
Error	185250.6	288	643.2312		
Total	1558951	324			

Table 2. Values of the parameter

Description	Parameter	
	MLM	MES
Makurdi	0.665	0.535
Abeokuta	0.695	0.478

Table 3. Results of ANOVA between the parameters obtained using the methods

Source of Variation	Sum of Square	Degree of freedom	Mean Sum of Square	F-Value	P-value
Between Parameters	0.030168	1	0.030168	29.06386	0.032727
Within Parameters	0.002076	2	0.001038		
Total	0.032244	3			

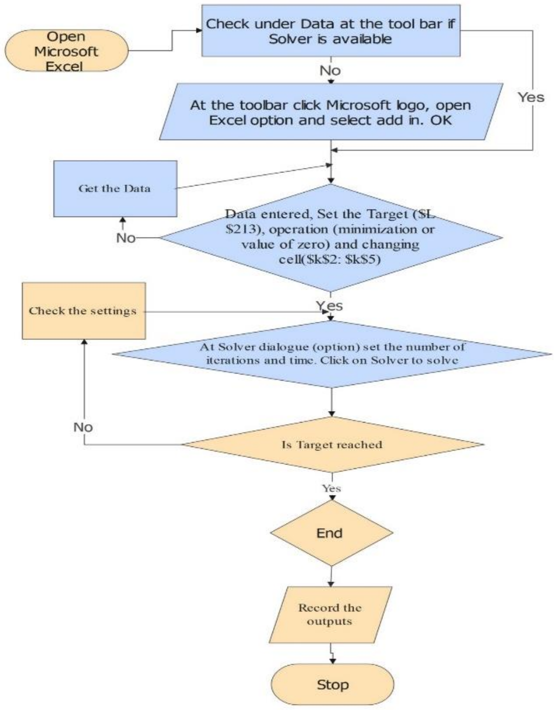

Fig. 4: Summary of the Microsoft Excel Solver procedures

Table 4. Results of ANOVA between the methods used (MLM and MES)

Source of Variation	Sum of Square	Degree of freedom	Mean Sum of Square	F-Value	P-value
Between Methods	0.000194	1	0.000194	0.01211	0.92242
Within Methods	0.03205	2	0.016025		
Total	0.032244	3			

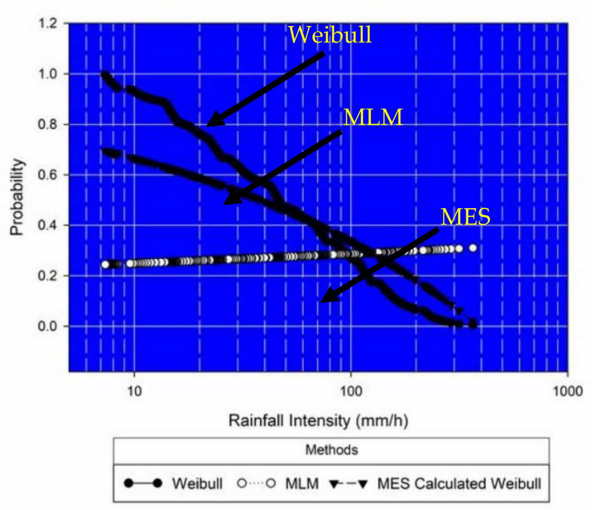

Fig. 5: Relationship between probabilities obtained using the methods (Abeokuta data)

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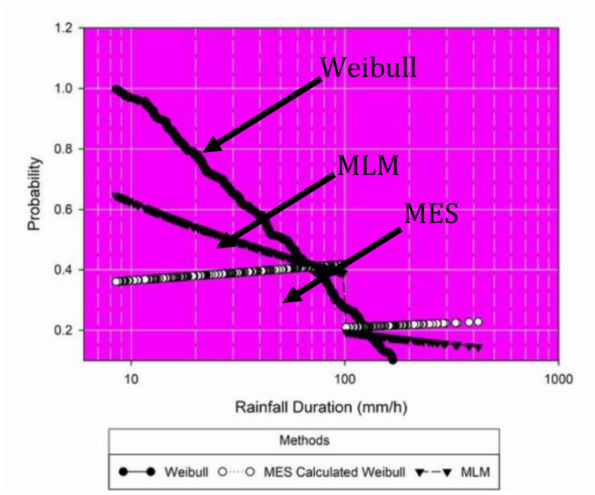

Fig. 6: Relationship between probabilities obtained using the methods (Makurdi data)

4 CONCLUSION

It was concluded based on the findings that MLM estimator was better than MES based on the values of MSC, CD, relative error and R. Also, MLM estimator predicted Bernoulli distribution of rainfall intensity better than MES.

REFERENCES

Akin, D, Sisiopiku, V.P and Skabardonis, A. (2011). Impacts of Weather on Traffic Flow Characteristics of Urban Freeways in Istanbul 6th International Symposium on Highway Capacity and Quality of Service Stockholm, Sweden June 28 – July 1, 2011. Procedia Social and Behavioral Sciences 16, 89–99

Al-Amri, N.S and Subyani, A.M (2017). Generation of Rainfall Intensity Duration Frequency (IDF) Curves for Ungauged Sites in Arid Region. Earth Syst Environ 1:8 DOI 10.1007/s41748-017-0008-8

Aleksandrovskaya L. N., Iosifov, P.A and Kirillin, A.V (2019). Analysis and comparison of some criteria of checking the probability distribution deviation from the normal distribution law. Int J Syst Assur Eng Manag. https://doi.org/10.1007/s13198-019-00761-5

Almarashi AM, Algarni A, and Nassar M (2020) On estimation procedures of stress-strength reliability for Weibull distribution with application. PLoS ONE 15(8): e0237997. https://doi.org/10.1371/journal.pone.0237997

Al-Zahrani, M.A. (2018). Assessing the impacts of rainfall intensity and urbanization on storm runoff in an arid catchment. Arab J Geosci 11, 208. https://doi.org/10.1007/s12517-018-3569-4

Baillon, J.-B., Cominetti, R., and Vaisman, J. (2015). A Sharp Uniform Bound for the Distribution of Sums of Bernoulli Trials. Combinatorics, Probability and Computing, 25(03), 352–361.

Balakrishnan, N., Koukouvinos, C., and Parpoula, C. (2014). On the Computation of Entropy Prior Complexity and Marginal Prior Distribution for the Bernoulli Model. Journal of Statistical Theory and Practice, 9(1), 59–72.

Barnett, T.S.; Grady, M, Singh, A.D and Purdy, K.G (2006). Combining Negative Binomial and Weibull Distributions for Yield and Reliability Prediction. IEEE Design & Test of Computers. 06, 0740-7475

Bianchi, F., Falsone, A., Prandini, M., and Piroddi, L. (2016). A randomised approach for NARX model identification based on a multivariate Bernoulli distribution. International Journal of Systems Science, 48(6), 1203–1216.

Brosius, J. (2015). An alternative method for the calculation of joint probability distributions. Application to the expectation of the triplet invariant. Acta Cryst.. A71, 76–81

Brun, X., and Melkote, S. N. (2012). Effect of Substrate Flexibility on the Pressure Distribution and Lifting Force Generated by a Bernoulli Gripper. Journal of Manufacturing Science and Engineering, 134(5), 051010.

Calisto, H., and Bologna, M. (2007). Exact probability distribution for the Bernoulli-Malthus-Verhulst model driven by a multiplicative colored noise. Physical Review E, 75(5), 234- 254.

Chacko, M and Mohan, R (2017). Bayesian analysis of Weibull distribution based on progressive type-II censored competing risks data with binomial removals. Computational Statistics. https://doi.org/10.1007/s00180-018-0847-2

Chai, L., Kong, L., Li, S., and Yi, W. (2020). The multiple model multi-Bernoulli filter based track-before-detect using a likelihood based adaptive birth distribution. Signal Processing, 171, 107501.

Chin, D. A. (2013) Water Resources Engineering, Prentice-Hall, Upper Saddle River, NJ.

Couton, F. ; Danech-Pajouh, M, and Broniatowski, M. (1997). Application of the mixture of probability distributions to the recognition of road traffic flow regimes. Transportation Systems, 715-720

Dai, B., Ding, S., and Wahba, G. (2013). Multivariate Bernoulli distribution. Bernoulli, 19(4), 1465–1483

David, A. O. Ify L. Nwaogazie and J. C. Agunwamba (2019). Modelling Rainfall Intensity by Optimization Technique in Abeokuta, South-West, Nigeria. Journal of Engineering Research and Reports 6(4): 1-10.

Denuit, M., and Vernic, R. (2018). Bivariate Bernoulli Weighted Sums and Distribution of Single-Period Tontine Benefits. Methodology and Computing in Applied Probability. doi:10.1007/s11009-018-9625-4

Du, H., and Xie, W. (2020). Extended Target Marginal Distribution Poisson Multi-Bernoulli Mixture Filter. Sensors, 20(18), 53-87. doi:10.3390/s20185387

Eggermont, P.P.B and LaRiccia, L.N (2009). Maximum Penalized Likelihood Estimation Volume II: Regression. Springer Series in Statistics. Springer Dordrecht Heidelberg

Fiondella, L., and Zeephongsekul, P. (2015). Trivariate Bernoulli distribution with application to software fault tolerance. Annals of Operations Research, 244(1), 241–255. doi:10.1007/s10479-015-1798-4

Gao, Z., Tao, J., Zhou, D., and Zeng, X. (2019). Efficient Parametric Yield Estimation over Multiple Process Corners via Bayesian Inference based on Bernoulli Distribution. IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems, 1–1. doi:10.1109/tcad.2019.2940682

Gharib, M., Ramadan, M. M., and Al-Ajmi, K. A. H. (2014). Some New Characterizations of Markov-Bernoulli Geometric Distribution Related to Random Sums. International Journal of Statistics and Probability, 3(3). doi:10.5539/ijsp.v3n3p138

Gharib. (2014). Characterization of Markov-Bernoulli Geometric Distribution Related To Random Sums. Journal of Mathematics and Statistics, 10(2), 186–191. doi:10.3844/jmssp.2014.186.191

Huber, M. (2016). A Bernoulli mean estimate with known relative error distribution. Random Structures and Algorithms, 50(2), 173–182.

Isikwue, M.O, Onoja, S.B and Laudan, K.J (2012). Establishment Of An Empirical Model That Correlates Rainfall-Intensity-Duration-Frequency For Makurdi Area, Nigeria. International Journal of Advances in Engineering and Technology, 5(1), 40-46

Jazi, M. A., Lai, C.-D., & Alamatsaz, M. H. (2010). A discrete inverse Weibull distribution and estimation of its parameters. Statistical Methodology, 7(2), 121–132. doi:10.1016/j.stamet.2009.11.001

Jeong, Y.-S., and Yon, Y.-H. (2020). A blockchain-based IoT data management scheme using Bernoulli distribution convergence in the mobile edge computing. Personal and Ubiquitous Computing. doi:10.1007/s00779-020-01459-3

Jeong, Y.-S., Shin, S.-S., and Han, K.-H. (2016). High-dimensionality priority selection scheme of bioinformatics information using Bernoulli distribution. Cluster Computing, 20(1), 539–546.

Jones, L.D. ; Vandeperrea, L.J. Haynesa,, T.A. and Wenman, M.R.(2020). Theory and application of Weibull distributions to 1D peridynamics for brittle solids. Comput. Methods Appl. Mech. Engrg. 363, 112903

Kadane, J. B. (2016). Sums of Possibly Associated Bernoulli Variables: The Conway–Maxwell-Binomial Distribution. Bayesian Analysis, 11(2), 403–420.

Kawakami, S., Sasaki, T., and Koashi, M. (2017). Finite-key analysis for quantum key distribution with weak coherent pulses based on Bernoulli sampling. Physical Review A, 96(1), 123 - 134. doi:10.1103/physreva.96.012305

Kawamura, K. (1988). The condition for an approximation of Poisson distribution to Bernoulli sums in multivariate distribution. Kodai Mathematical Journal, 11(2), 280–286.

Kazuki, S.;, Masato, K., Ken-Ichi, O.; and Hirotaka, T. (2017), Probability Distributions of means of IA and IF for Gaussian noise and its application to an anomaly detection, Advances in Data Science and Adaptive Analysis, doi: 10.1142/S2424922X18500067

Kullback, S. (1935). On the Bernoulli distribution. Bulletin of the American Mathematical Society, 41(12), 857–865

Kwak, T.-Y., Woo, S.-I., Chung, C.-K., and Kim, J.:(2020) Experimental assessment of the relationship between rainfall intensity and sinkholes caused by damaged sewer pipes, Nat. Hazards Earth Syst. Sci., 20, 3343–3359, https://doi.org/10.5194/nhess-20-3343-2020,.

Lidiya N. Aleksandrovskaya, L.N; Ardalionova, A.E and Papic, L. Application of probability distributions mixture of safety indicator in risk assessment problems. Int J Syst Assur Eng Manag. https://doi.org/10.1007/s13198-019-00760-6

Luo, J., Tian, W., Zhong, S., Shi, K., and Liao, D. (2019). Non-fragile asynchronous reliable sampled-data control for uncertain fuzzy systems with Bernoulli distribution. Journal of the Franklin Institute. doi:10.1016/j.jfranklin.2019.10.

Luo, J., Tian, W., Zhong, S., Shi, K., Chen, H., Gu, X.-M., and Wang, W. (2017). Non-fragile asynchronous H ∞ control for uncertain stochastic memory systems with Bernoulli distribution. Applied Mathematics and Computation, 312, 109–128

Molotkov, S. N. (2018). Quantum Key Distribution As a Scheme with Bernoulli Tests. Journal of Experimental and Theoretical Physics, 126(6), 741–752

Oke, I. A (2008) Utilization of Weibull Techniques For Short Term Data In Environmental Engineering. Environmental Engineering and Sciences. 25 (7),1099-1107.

Oke, I. A; Umoru, L.E; Oladepo, K.T and Ogedengbe, M.O (2008) Utilization Of Weibull Techniques To Describe Stability Distribution Of Carbon Resin Electrodes. Ife Journal of Technology, 17 (1), 35-46.

Parisa H., P., Tabaria, H and Willems, P (2020). Climate change impact on short-duration extreme precipitation and intensity–duration–frequency curves over Europe. Journal of Hydrology 590, 125 - 149

Pham, H., and Pham, D. H. (2019). A Median-Based Machine-Learning Approach for Predicting Random Sampling Bernoulli Distribution Parameter. Vietnam Journal of Computer Science, 06(01), 17–28

Piast, R. W., and Piast, R. (2019). Shannon's Information, Bernal's Biopoiesis and Bernoulli Distribution as Pillars for Building a Definition of Life. Journal of Theoretical Biology. doi:10.1016/j.jtbi.2019.03.009

Pichon, F. (2018). Canonical decomposition of belief functions based on Teugels' representation of the multivariate Bernoulli distribution. Information Sciences, 428, 76–104.

Pichugina, S. V. (2008). Application of the Theory of Truncated Probability Distributions to Studying Minimal River Runoff: Normal and Gamma Distributions. Water Resources, 35(1), 23–29.

Pieracci, A. (1995). Parameter estimation for Weibull probability distribution function of initial fatigue quality. AIAA Journal, 33(9), 1574–1581. doi:10.2514/3.12858

Pisarenko, V. F. Bolgov, M. V. Osipova,, N. V. and Rukavishnikova, T. A. (2002). Application of the Theory of Extreme Events to Problems of Approximating Probability Distributions of Water Flow Peaks. Water Resources, 29(6), 593–604.

Pobočíková, I.; Sedliačková, Z. and Michalková, M (2017). Application of four probability distributions for wind speed modelling. TRANSCOM 2017: International scientific conference on sustainable, modern and safe transport. Procedia Engineering 192, 713 – 718

Rinne, H. (2009). The Weibull Distribution A Handbook Chapman & Hall/CRCTaylor & Francis Group6000 Broken Sound Parkway NW, Suite 300 Boca Raton, FL 33487-2742

Roberts, L. A. (2019). Distribution free goodness of fit testing of grouped Bernoulli trials. Statistics and Probability Letters. doi:10.1016/j.spl.2019.01.035

Santos, L. S. F. (2018). The energy density distribution of an ideal gas and Bernoulli's equations. European Journal of Physics, 39(3), 035102. doi:10.1088/1361-6404/aaa34c

Sulbhewar, L. N., and Raveendranath, P. (2017). Performance of consistent through-thickness electric potential distribution for Euler-Bernoulli piezoelectric beam finite elements. International Journal of Computer Aided Engineering and Technology, 9(2), 179 - 187.

Telles, C. R. (2020). Measuring nonlinearity by means of static parameters in Bernoulli binary sequences distribution: a brief approach. International Journal of Modeling, Simulation, and Scientific Computing. doi:10.1142/s179396232050021x

Teyabeen, A.A; Akkari, F.R and Jwaid, A.E (2017). Comparison of Seven Numerical Methods For Estimating Weibull Parameters For Wind Energy Applications. 2017 UK Sim-AMSS 19th International Conference on Modelling & Simulation, 173-180. DOI 10.1109/UKSim.2017.31

Ward, M.D and Ahlquist, J.S (2018). Maximum Likelihood for Social Science Strategies for Analysis. University Printing House, Cambridge CB2 8BS, United Kingdom.

Wentzel-Larsen, T., and Anhøj, J. (2019). Joint distribution for number of crossings and longest run in independent Bernoulli observations. The R package cross run. PLOS ONE, 14(10), e0223233. doi:10.1371/journal.pone.0223233

Yashunsky, A. D. (2019). Limit Points of Bernoulli Distribution Algebras Induced by Boolean Functions. Lobachevskii Journal of Mathematics, 40(9), 1423–1432.

Zchaluk, K., and Foster, D. H. (2009). Model-free estimation of the psychometric function. Attention, Perception, & Psychophysics, 71(6), 1414–1425. doi:10.3758/app.71.6.14

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